Quantum Deformations of Conformal Algebras Introducing Fundamental Mass Parameters

Jerzy Lukierski *Scientific †, Pierre Minnaert and Marek Mozrzymas *
Laboratoire de Physique Théorique,
Université Bordeaux I,
33175 Gradignan, France

Abstract

We consider new class of classical r-matrices for $D = 3$ and $D = 4$ conformal Lie algebras. These r-matrices do satisfy the classical Yang-Baxter equation and as two-tensors belong to the tensor product of Borel subalgebra. In such a way we generalize the lowest order of known nonstandard quantum deformation of $sl(2)$ to the Lie algebras $sp(4) \cong so(5)$ and $sl(4) \cong so(6)$. As an exercise we interpret non-standard deformation of $sl(2)$ as describing quantum $D = 1$ conformal algebra with fundamental mass parameter. Further we describe the $D = 3$ and $D = 4$ conformal bialgebras with deformation parameters equal to the inverse of fundamental masses. It appears that for $D = 4$ the deformation of the Poincaré algebra sector coincides with "null plane" quantum Poincaré algebra.

CPTMB/PT/95-6 June 1995

*On leave of absence from the Institute of Theoretical Physics, University of Wrocław, 50-204 Wrocław, Poland.

†Partially supported by KBN grant 2P 302 087 06.

1 Introduction

Recently, quantum deformations of $D = 4$ space-time symmetries were considered (see e.g. [1–16]). Because the Poincaré generators (P_μ, $M_{\mu\nu}$) transform nontrivially under the change of length scale

$$P'_m = \lambda^{-1} P_\mu \, , \qquad M'_{\mu\nu} = M_{\mu\nu} \, , \tag{1.1}$$

in the case of $D = 4$ Poincaré algebra one can distinguish two different types of quantum deformations:

a) With dimensionless deformation parameter q [2,4,12,15].

   In such a case the deformed Poincaré algebra $U_q(P_4)$ is invariant under the rescaling (1.1) with the value of q kept fixed.

b) With dimensionfull mass-like parameter κ [1,3-5,7-9,13,14].

   The κ-deformed Poincaré algebra, denoted by $U_\kappa(P_4)$ is invariant under rescaling (1.1) provided that we rescale also $\kappa \longrightarrow \kappa' = \lambda^{-1}\kappa$. Such a deformation occurs naturally in the bicrossproduct description of quantum Poincaré algebra [8], with the deformed semidirect product of the Lorentz algebra and fourmomentum sector containing nonlinear functions of the fourmomenta. The scale invariance of these nonlinear functions under the transformation (1.1) leads necessarily to the appearence of a mass-like deformation parameter.

In this paper we shall consider the quantum deformations of the conformal algebras with a dimensionfull deformation parameter. The importance of this problem can be justified as follows:

i) The conformal symmetry is the fundamental "master" symmetry of space-time, containing two other fundamental geometries (Poincaré , de Sitter) as its broken cases.

ii) The conformal symmetries describe the world of massless particles and fields. Usually, masses breaking conformal symmetries are introduced on the level of field representations of the symmetry group. The introduction of the mass-like deformation parameter leads to the appearence of the fundamental mass on basic geometrical level.

iii) From the mathematical point of view, the deformations of conformal algebras with fundamental mass parameter introduce a new type of quantum deformations of Lie algebras, generalizing the nonstandard deformation of $sl(2)$ [17-21].

The quantum deformations of Lie algebras are described by the noncommutative and noncocommutative Hopf algebras, which can be obtained by the quantization of corresponding Lie bialgebras [22, 23]. For semisimple Lie algebras the classification of quantum deformations is provided by the choice of classical r-matrices, determining the cocommutators of Lie bialgebra. In particular the classical Yang-Baxter equations is required if we

assume that after completing the quantum deformation procedure we get the quasitriangular Hopf algebra with quantum universal R-matrix, satisfying quantum Yang-Baxter equation [17].

In this paper we shall consider the choices of classical r-matrices, generalizing for $sp(4) \cong so(5)$ and $sl(4) \cong so(6)$ the nonstandard deformation of $sl(2)$ [17-21], described by the classical r-matrices $r_{\pm} = h \wedge e_{\pm}$ ($A \wedge B \equiv (A \otimes B - B \otimes A)$).

The plan of this paper is the following:

In Sect. 2 we recall a known example of nonstandard deformation of $sl(2)$ Lie algebra and we interprete it as the quantum deformation od $D = 1$ conformal algebra introducing mass-like deformation parameter.

In Sect. 3 we introduce for any simple Lie algebra $\hat{g}$ the class of nonstandard classical r-matrices described by two-tensors on its Borel subalgebra $\hat{b}$. In particular we shall be interested in maximal nonstandard classical r-matrices of $\hat{g}$, i.e. those which cannot be described by the generators of any simple subalgebra $\hat{g}' \subset \hat{g}$ ($\hat{g}' \neq \hat{g}$). We present these maximal nonstandard classical r-matrices in Cartan-Weyl basis for the complex Lie algebras $so(5) \cong sp(4)$ and $so(6) \cong sl(4)$; further we shall impose the $so(3,2)$ and $so(4,2)$ reality conditions.

In Sect. 4 and 5 we introduce the physical (conformal) generators for $so(3,2)$ ($D = 3$ conformal) and $so(4,2)$ ($D = 4$ conformal). It appears that the nonstandard r-matrices considered in sec. 3 lead to the introduction of the deformation parameters $\frac{1}{M_i}$ (M_i - fundamental masses). More specifically we obtain that

i) For $D = 3$ the deformation is described by two masses M_1, M_2. If $M_2 = 0$ our classical r-matrix generates the quantum defomation of $D = 3$ Poincaré algebra.

ii) For $D = 4$ conformal algebra the deformation generated by our maximal nonstandard r-matrix introduces one fundamental mass M and describes the defomation of the Poincaré subalgebra with classical fourdimensional subalgebra $e(2) \oplus r$ ($e(2)$ is $D = 2$ inhomogenous Euclidean algebra and r generator of noncompact Abelian group R). It is interesting that such a deformation has been obtained recently in [24] under the name of "null-plane" quantum Poincaré algebra in different context - by so-called deformation embedding method [25] from three to four dimensions.

In Sect. 6 we present the discussion and outlook.

2 Quantum deformations of $D = 1$ conformal algebra with mass- like deformation parameter

It is well known that the $D = 1$ conformal algebra

$$[D, P] = P\,, \qquad [D, K] = -K\,, \qquad [P, K] = 2D\,, \tag{2.1}$$

where P describes the time translations (energy), D — the scaling transformations, and K — the conformal accelerations, after the identification

$$e_+ = P\,, \qquad e_- = K\,, \qquad h = 2D\,, \tag{2.2}$$

can be written as the $sl(2,R) \simeq O(2,1)$ algebra in Cartan- Chevalley basis

$$[h, e_\pm] = \pm 2 e_\pm \,, \qquad [e_+, e_-] = h \,. \tag{2.3}$$

For $sl(2,R)$ there exist only two inequivalent deformations, generated by the classical r-matrices with the following antisymmetric parts:
i)standard deformation [22–24]

$$r_s = c_s e_+ \wedge e_- \,. \tag{2.4}$$

Adding suitable symmetric part one gets from (2.4) the linear term in the coproduct of Drinfeld-Jimbo quantum algebra $U_q(sl(2))$. It is easy to see that the invariance of (2.4) under the rescalings $P \longrightarrow \lambda^{-1} P$, $K \longrightarrow \lambda K$ imply that the deformation parameter c_s is dimensionless.
ii)nonstandard deformations [17-21].

$$r_\pm = c_\pm h \wedge e_\pm \qquad (c_\pm \text{ — constants}). \tag{2.5}$$

Using (2.2) one obtains

$$r_+ = 2c_+ D \wedge P \,, \qquad r_- = 2c_- D \wedge K \,. \tag{2.6}$$

We see that the quantum deformation generated by the classical r- matrix r_+ provides the deformation parameter transforming as the inverse of mass ($2c_+ = \frac{1}{M}$; we shall call it M-deformation) and r_- implies that $2c_- = \widetilde{M}$ ($\widetilde{M}$-deformation), where M, $\widetilde{M}$ are fundamental masses.
The quantization of the Lie algebra $sl(2,R)$ generated by r_+ has been given firstly by Ohn [21]. The relations (2.1) in all orders of $\frac{1}{M}$ are deformed as follows:

$$\begin{aligned} [D,P] &= M \sinh \tfrac{P}{M} \,, \qquad [P,K] \;=\; 2D \,, \\ [D,K] &= \tfrac{1}{2}(-K \cosh \tfrac{P}{M} - (\cosh \tfrac{P}{M})K) \,. \end{aligned} \tag{2.7}$$

The coproduct and the antipode take the form:

$$\begin{aligned} \Delta(P) &= P \otimes 1 + 1 \otimes P \,, \\ \Delta(D) &= e^{-\frac{P}{M}} \otimes D + D \otimes e^{\frac{P}{M}} \,, \\ \Delta(K) &= e^{-\frac{P}{M}} \otimes K + K \otimes e^{\frac{P}{M}} \,, \end{aligned} \tag{2.8}$$

$$S(P) = -P, \qquad S(K) \;= -K - \frac{1}{M}(D - \sinh \frac{P}{M}) \qquad S(D) \;= -D - 2\sinh \frac{P}{M}. \tag{2.9}$$

We would like to make the following remarks:

i) The quantum deformation (2.7)-(2.8) has the $D=1$ quantum Weyl Hopf subalgebra span by two generators P and D.

ii) The deformation generated by the classical r-matrix r_- (see (2.5)-(2.6)) is obtained by the following replacement in the formulae (2.7)-(2.9)

$$\begin{gathered} P \longrightarrow K\,, \qquad K \longrightarrow P\,, \qquad D \longrightarrow -D\,, \\ M \longrightarrow \tfrac{1}{\widetilde{M}}\,. \end{gathered} \tag{2.10}$$

It appears that the M-deformation is described by the power series in the variable M^{-1} (like the κ-deformation of Poincaré algebra) and the $\widetilde{M}$-deformation — as the power series in $\widetilde{M}$. Respectively, the "classical" no-deformation limits are $M \longrightarrow \infty$ and $\widetilde{M} \longrightarrow 0$.

iii) Because $D = 2$ conformal algebra $so(2,2) \simeq so(2,1) \oplus so(2,1)$, the deformation of $D = 2$ conformal algebra is determined by the pair of (standard or nonstandard) deformations of $so(2,1) \cong sl(2,R)$. In fact the decomposition of the conformal algebra into one-dimensional conformal algebras of right- movers and left-movers permits to introduce different M and $\widetilde{M}$ parameters in these two chiral sectors. The description of quantum $D = 2$ conformal algebra by a pair of nonstandard deformations with one parameter was proposed recently in [26]. The application of nonstandard deformation of $so(2,1)$ to the description of quantum-mechanical model is also under consideration [27].

3 Nonstandard classical r-matrices for $sp(4;C)$, $sl(4;C)$ and their real forms

We shall consider the nonstandard deformations of a simple Lie algebra $\hat{g}$ generated by classical $r_\pm$ -matrices satisfying the following two conditions:

a) $r_\pm \in \hat{b}_\pm \wedge \hat{b}_\pm$, where $\hat{b}_\pm$ describes the Borel subalgebras, span respectively by (h_i, $e_{\pm a}$) ($i = 1 \ldots r = \text{rank}\hat{g}$, $a = 1 \ldots N = \frac{1}{2}(\text{dim}\hat{g} - r)$), where h_i describes the Abelian Cartan generators, and $e_{+a}(e_{-a})$the positive (negative) root generators.

b) $r_\pm$ satisfy the classical Yang-Baxter equation

$$[r_{\pm 12}, r_{\pm 13}] + [r_{\pm 12}, r_{\pm 23}] + [r_{\pm 13}, r_{\pm 23}] = 0\,, \tag{3.1}$$

where if $r = r^{AB} I_A \otimes I_B$ (I_A – Cartan-Weyl basis of $\hat{g}$; $A, B = 1 \ldots \text{dim}g$), $r_{12} = r^{AB} I_A \otimes I_B \otimes 1$ etc.

We shall introduce also the notion of maximal nonstandard r-matrix, belonging to the class described above. Let us observe that if $\hat{g}' \subset \hat{g}$ ($\hat{g}'$ is a simple Lie subalgebra of $\hat{g}$), then any classical r-matrix for $\hat{g}'$ is also a classical r-matrix for $\hat{g}$. For the maximal nonstandard classical r-matrices such an embedding does not exist - the algebra $\hat{g}$ is the minimal simple algebra, providing Borel algebras $\hat{b}_\pm$ in the formula $r_\pm \in \hat{b}_\pm \wedge \hat{b}_\pm$. Below

we shall consider only the maximal nonstandard classical r- matrices $r_+ \in \hat{b}_+ \wedge \hat{b}_+$. It appears that the matrices $r_- \in \hat{b}_- \wedge \hat{b}_-$ can be obtained by suitable anti- automorphism of $\hat{g}$.

In physical applications there are relevant real Lie algebras. We shall consider therefore only the classical r-matrices satisfying the reality condition:

$$(r_+)^+ = ((r_+)^{AB} I_A \wedge I_B)^+ = ((r_+)^{AB})^* I_A^+ \wedge I_B^+ = r_+ \tag{3.2}$$

We assume that $+$–involution is an anti-automorphism of $U(\hat{g})$ (i.e. $(ab)^+ = b^+ a^+$) and automorphism of tensor product (i.e. $(a \otimes b)^+ = a^+ \otimes b^+$). Because from (3.2) follows that $(b_+)^+ \subset b_+$, the involutions implying (3.2) map $\Delta^\pm \longrightarrow \Delta^\pm$ where by $\Delta^+(\Delta^-)$ we denote the set of positive (negative) root generators. It appears that due to the parabolic decomposition of the conformal algebra $so(D,2)$ (see e.g. [4]) indeed such a type of reality conditions describe real conformal algebras.

The general framework describing solutions of the classical Yang- Baxter equation, related to the Borel subalgebras of simple Lie algebras was considered in [28] (see also [29]). Below we shall describe two examples:

a) $Sp(4,C) \cong so(5,C)$ ($D = 3$ complex conformal algebra)
   The Borel subalgebra b_+ has the basis $(h_1, h_2, e_1, e_2, e_3, e_4)$ where e_1, e_2, are the simple root generators (Cartan-Chevalley basis) and

$$e_3 = [e_1, e_2], \qquad e_4 = [e_1, e_3], \tag{3.3}$$

   The maximal nonstandard classical r-matrix takes the form

$$r_+ = c_3^{(1)}(h_1 \wedge e_4 - e_1 \wedge e_3) + c_3^{(2)} h_2 \wedge e_4 \tag{3.4}$$

   The $+$-involution introducing the real form $Sp(4,R) \cong so(3,2)$ we choose the following (see e. g. [30]; $i = 1,2$):

$$\begin{aligned} h_i^+ &= -h_i\,, & & \\ e_1^+ &= \lambda e_1\,, & e_2^+ &= \epsilon e_2\,, \\ e_3^+ &= -\lambda\epsilon e_3\,, & e_4^+ &= \epsilon e_4\,, \end{aligned} \tag{3.5}$$

   where $\lambda^2 = \epsilon^2 = 1$.
   From the invariance of (3.4) under (3.5) follows that $\epsilon = \pm 1, \lambda = \pm 1$. One gets for real $c_3^{(1)}$, $c_3^{(2)}$ that $\epsilon = -1$, $\lambda = \pm 1$. It can be seen from [31] that both real forms describe the real conformal algebra $so(3,2)$.

b) $sl(4) \cong so(6,C)$ ($D = 4$ complex conformal algebra).
   The simple root generators $e_{\pm i}$ ($i = 1,2,3$) and Cartan generators h_i define the remaining part of Cartan-Weyl basis as follows:

$$\begin{aligned} e_4 &= [e_1, e_2]\,, & e_{-4} &= [e_{-2}, e_{-1}]\,, \\ e_5 &= [e_2, e_3]\,, & e_{-5} &= [e_{-3}, e_{-2}]\,, \\ e_6 &= [e_1, e_5]\,, & e_{-6} &= [e_{-5}, e_{-1}]\,, \end{aligned} \tag{3.6}$$

and

$$h_4 = h_1 + h_2\,, \qquad h_5 = h_2 + h_3\,, \qquad h_6 = h_1 + h_2 + h_3\,. \tag{3.7}$$

The $sl(4)$ Lie algebra can be written compactly as follows $(A, B = 1, \ldots 6)$

$$\begin{aligned} [h_A, e_{\pm B}] &= \pm\alpha_{AB} e_{\pm B} \qquad \text{(no summation)}\,, \\ [e_A, e_{-B}] &= \delta_{AB} h_B \qquad \text{(no summation)}\,, \end{aligned} \tag{3.8}$$

where the extended symmetric Cartan matrix is given by the formula (see e.g. [32])

$$\alpha_{AB} = \begin{pmatrix} 2 & -1 & 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \tag{3.9}$$

The remaining $sl(4)$ Lie algebra relations (besides the ones given by (3.6)) are obtained from Serre relations for the generators $e_{\pm i}$ $(i = 1, 2, 3)$ and take the form:

$$\begin{aligned} [e_1, e_3] &= [e_1, e_4] = [e_1, e_6] = 0\,, \\ [e_2, e_4] &= [e_2, e_5] = [e_2, e_6] = 0\,, \\ [e_3, e_4] &= e_6\,, \qquad [e_3, e_5] = [e_3, e_6] = 0\,, \\ [e_4, e_5] &= [e_4, e_6] = [e_5, e_6] = 0\,. \end{aligned} \tag{3.10}$$

For $sl(4, C)$ we found the following maximal nonstandard classical r- matrix

$$r_+ = c_4^{(1)}(h_1 - h_3) \wedge e_6 + c_4^{(2)}(h_3 \wedge e_6 + e_1 \wedge e_5 - e_3 \wedge e_4) \tag{3.11}$$

From the discussion of all real forms for $sl(4, C)$ (see e.g. [32,33] and put $q = 1$) we select one which maps $\Delta^\pm \longrightarrow \Delta^\pm$ and describes the conformal algebra $so(4, 2)$:

$$\begin{aligned} h_1^+ &= -h_3\,, & h_2^+ &= -h_2\,, \\ e_1^+ &= \epsilon e_3\,, & e_2^+ &= \eta e_2\,, \\ e_4^+ &= \eta\epsilon e_5\,, & e_6^+ &= \eta e_6\,, \end{aligned} \tag{3.12}$$

From (3.11-12) follows that the reality conditions (3.12) imply $\eta = \pm 1, \epsilon = \pm 1$. For $\eta = -1$, $\epsilon = \pm 1$ we choose $c_4 = c_4^{(2)} = 2c_4^{(1)}$ (c_4 real).

4 $\quad D = 3$ Conformal Algebra

For the description of $D = 3$ conformal algebra in terms of $Sp(4) \cong so(5)$ Cartan-Weyl basis satisfying the reality conditions (3.5) we shall use the formulae given in [30] (we put

for them $q = 1$). Selecting for simplicity $\lambda = -\epsilon = 1$ we obtain

$$\begin{aligned} h_1 &= M_{12}\,, & h_2 &= M_{04} - M_{12}\,, \\ e_1 &= \tfrac{1}{\sqrt{2}}(M_{23} + M_{31})\,, & e_3 &= \tfrac{1}{\sqrt{2}}(M_{03} + M_{34}) \end{aligned} \tag{4. 1a}$$

$$\begin{aligned} e_2 &= -\tfrac{1}{\sqrt{2}}(M_{14} + M_{24} + M_{01} + M_{02})\,, \\ e_4 &= \tfrac{1}{\sqrt{2}}(M_{14} + M_{01} - M_{24} - M_{02})\,. \end{aligned} \tag{4. 1b}$$

where $M^\dagger_{AB} = -M_{AB} = M_{BA}$ $(A, B = 0, 1, 2, 3, 4)$, and it follows that [30,31]

$$[M_{AB}, M_{CD}] = \eta_{BC} M_{AD} + \eta_{AD} M_{BC} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}\,, \tag{4.2}$$

where $\eta_{AB} = \mathrm{diag}(-1, 1, -1, 1, 1,)$. The generators $M_{31} = J, M_{12} = L_1, M_{23} = L_2$ form $D = 3$ Lorentz algebra $so(2, 1)$. The rest of conformal generators are defined as follows:

$$\begin{aligned} M_{01} &= \tfrac{1}{\sqrt{2}}(P_1 + K_1)\,, & M_{41} &= -\tfrac{1}{\sqrt{2}}(P_1 - K_1)\,, \\ M_{02} &= \tfrac{1}{\sqrt{2}}(P_0 + K_0)\,, & M_{42} &= -\tfrac{1}{\sqrt{2}}(P_0 - K_0)\,, \\ M_{03} &= \tfrac{1}{\sqrt{2}}(P_2 + K_0)\,, & M_{43} &= -\tfrac{1}{\sqrt{2}}(P_2 - K_0)\,, \end{aligned} \tag{4. 3a}$$

$$D = M_{04}\,. \tag{4. 3b}$$

Substituting this physical basis into the formulae (4.1) one obtains that ($P_\pm = P_0 \pm P_1$)

$$r_+ = \frac{1}{M_1}(L_1 \wedge P_1 - (L_2 + J) \wedge P_2) + \frac{1}{M_2}(D - L_1) \wedge P_- \tag{4.4}$$

where$M_1 = 2(c_3^{(1)})^{-1}, M_2 = 2(c_3^{(2)})^{-1}$ describe the fundamental mass parameters. One can make the following remarks:

i) If $M_2 = \infty, M_1 < \infty$, the classical r-matrix (4.4) describes the deformation of $D = 3$ Poincaré algebra. This deformation is different from the κ-deformation of $D = 3$ Poincaré algebra described by the following classical r-matrix [13]

$$r_+ = \frac{1}{\kappa}(L_1 \wedge P_1 + L_2 \wedge P_2) \tag{4.5}$$

ii) Two other selected cases are $M_1 = M_2$ and $M_1 = \infty$. The second case describes due to relation $[D - L_1, P_-] = 0$ so called soft deformations [34].

5 $D = 4$ Conformal Algebra

We shall express the $sl(4, C)$ Cartan-Weyl generators satisfying the reality conditions (3.12) by using the formulae given in [32]. The relations (3.8) and (3.10) imply $so(4, 2)$ classical $D = 4$ conformal algebra relations ($M^\dagger_{KL} = -M_{KL} = M_{LK}$; K, L =0, 1, 2, 3, 4, 5)

$$[M_{KL}, M_{PR}] = g_{KR} M_{LP} + g_{LP} M_{KR} - g_{LR} M_{KP} - g_{KP} M_{LR}\,, \tag{5.1}$$

where $g_{AB} = \mathrm{diag}(-1,1,1,1,1,-1)$ and

$$\begin{array}{rclrclrcl} P_\mu &=& (M_{4\mu}+M_{5\mu})\,, & K_\mu &=& (M_{5\mu}-M_{4\mu})\,, & & & \\ M_i &=& \frac{1}{2}\epsilon_{ijk}M_{jk}\,, & L_i &=& M_{0i}\,, & D &=& M_{45} \end{array} \tag{5.2}$$

Introducing $M_\pm = M_1 \pm iM_2, L_\pm = L_1 \pm iL_2$ one gets (we choose $\epsilon = 1$ in (3.12))

$$\begin{array}{ll} h_1{=}L_3 - iM_3)\,, & h_3{=}L_3 + iM_3\,, \\ e_1{=}\frac{1}{2}(M_+ + iL_+)\,, & e_3{=}-\frac{1}{2}(M_- - iL_-)\,, \\ e_2{=}\frac{1}{2}(P_0 - P_3)\,, & e_6{=}\frac{1}{2}(P_0 + P_3)\,, \\ e_4{=}\frac{i}{2}(P_1 + iP_2)\,, & e_5{=}-\frac{i}{2}(P_1 - iP_2)\,, \end{array} \tag{5. 3a}$$

$$h_2 = -(D + L_3)\,. \tag{5. 3b}$$

Substituting this physical basis into the formula (3.11) with $c_4 = c_4^{(2)} = 2c_4^{(1)} = \frac{2}{M}$ one obtains ($P_+ = P_0 + P_3$)

$$r_+ = \frac{1}{M}[L_3 \wedge P_+ + (M_2 + L_1) \wedge P_1 - (M_1 - L_2) \wedge P_2] \tag{5.4}$$

The algebra $\widetilde{e}(2)$ with the generators

$$\widetilde{E}_1 = L_1 + M_2, \qquad \widetilde{E}_2 = -L_2 + M_1, \qquad \widetilde{E}_3 = L_3, \tag{5.5}$$

has the following commutation relations

$$[\widetilde{E}_1, \widetilde{E}_2] = 0, \qquad [\widetilde{E}_1, \widetilde{E}_3] = -\widetilde{E}_1, \qquad [\widetilde{E}_2, \widetilde{E}_3] = -\widetilde{E}_2 \tag{5.6}$$

and describes $D = 2$ Poincaré algebra.
One can decompose the Lorentz algebra $so(3,1)$ as follows

$$so(3,1) = \widetilde{e}(2) \oplus e(2) \tag{5.7}$$

where $e(2)$ generators are the following

$$E_1 = L_1 - M_2, \qquad E_2 = L_2 + M_1, \qquad E_3 = M_3, \tag{5.8}$$

and satisfy the $D = 2$ Euclidean algebra relations

$$[E_1, E_2] = 0, \qquad [E_2, E_3] = -E_1, \qquad [E_1, E_3] = E_2, \tag{5.9}$$

The M-deformation, generated by the classical r-matrix (5.4) leads to the quantum deformation of the Poincaré algebra obtained recently in [24] by different method. It is easy to check that the classical r-matrix (5.4) does not modify the coproducts for the generators $(M_3, \widetilde{E}_1, \widetilde{E}_2)$, forming another $D = 2$ Euclidean algebra as well as the component P_+ ($P_+ = P_0 + P_3$) of the four-momentum. The fourdimensional algebra with the generators

$(M_3, \widetilde{E}_1, \widetilde{E}_2, P_+)$ describe the classical subalgebra of the M-deformed $D = 4$ coformal algebra. The mass Casimir is given by the formula [24]

$$C_2 = P_1^2 + P_2^2 - MP_- sinh(\frac{P_+}{M}) \tag{5.10}$$

Because the deformation $U_M(\mathcal{P}_4)$ of the Poincaré algebra, called in [24] the "null plane" quantum Poincaré algebra forms a Hopf subalgebra of $U_M(o(4,2))$, it is the deformed mass-shell (5.10) which describes the deformation of massless d'Alembert operator in a way which permits also to introduce the κ-deformed conformal properties. Having the formula for the classical r-matrix one can look for the quantum deformations of the generators K_μ and D using e.g. the perturbative formulae for quantisation of bialgebras given by Drinfeld [35] and Reshetikhin [36]. We conjecture that similarly as the relation $\vec{P}^2 - P_0^2 = 0$ is conformal invariant in undeformed theory, we shall obtain that for $M < \infty$

$$[C_2, K_\mu] \mid_{C_2=0} = [C_2, D] \mid_{C_2=0} = 0 \tag{5.11}$$

The proof of relation (5.11) would additionally justify the choice of the deformation generated by classical r-matrix (5.4) as the most appropriate for the massless conformal-invariant theories and explain the difficulties with embedding of κ-Poincaré algebra $U_\kappa(\mathcal{P}_4)$ into the quantum conformal algebra (see [37]).

6 Discussion and Outlook

In this paper we proposed new deformation schemes for $D = 3$ and $D = 4$ conformal algebras introducing fundamental mass parameters. The existence of a fundamental scale as a lower bound to any position measurement seems to be a consequence of quantisation of general relativity (see e. g. [38]). The considerations of this paper introduce on purely geometric basis the notion of deformed conformal structures, which should modify the classical notions of distance and causality for short distances. By suitable adjustment of the mass-like deformation parameters one should be able to restrict such a modifications to the distances comparable or shorter than the Planck length ($\Delta l \simeq 10^{-33} cm$).
The deformations described by classical r-matrices (4.4) and (5.4) describe new $D = 3$ and $D = 4$ conformal bialgebras. If we introduce the fundamental matrix realizations of $D = 3$ and $D = 4$ classical conformal algebras (4×4 real matrices for $D = 3$ and 4×4 complex matrices for $D = 4$) one can write down the Poisson brackets describing the Lie-Poisson structure on $D = 3$ and $D = 4$ conformal groups [22].Following the derivation in [13] of $D = 4$ quantum Poincaré group one can quantize the Poisson-Lie brackets and obtain quantum $D = 3$ and $D = 4$ conformal groups.

We would like to recall here that the κ-deformation of Poincaré algebra (see [1, 3, 8]) leads to classical nonrelativistic symmetries $so(3) \subset so(3,1)$ but the "null plane" quantum Poincaré algebra given in [23] provides classical symmetries for the subgroup

$e(2) \subset so(3,1)$. The subgroups $so(3)$ and $e(2)$ are respectively the stability groups for the time-like and light-like four-momenta. One can also introduce the third deformation, leading to the classical symmetries for the subalgebra $so(2,1) \subset so(3,1)$ not changing the components of the space-like (tachyonic) four-momenta, by assuming the following classical r-matrix (see [39])

$$r = M_1 \wedge P_2 - M_2 \wedge P_1 + L_3 \wedge P_0 \tag{6.1}$$

where we have chosen as the "deformation direction" the third space axis. It is quite possible that <u>different</u> quantum deformations of the $D = 4$ Poincaré algebra are appropriate for the world with matter propagating with sublumimal velocities, light velocities and superluminal velocites. These three cases can be considered in a nice geometric setting if we introduce the quantum deformation of the Poincaré algebra as the bicrossproduct Hopf algebra [8]:

$$U(\mathcal{P}_4) = U(so(3,1) \bowtie T_4^\kappa \tag{6.2}$$

The three classes of deformations discussed above are defined by the choice in (6.2) of the part of the Lorentz algebra $so(3,1)$ remaining unaffected by the κ-deformed action of $so(3,1)$ generators on the four-momentum sector T_4^κ. This way of looking at different deformations of Poincaré algebra is under considerations.

Acknowledgement
One of the author (J. L.) would like to thank C. Gomez, O. Ogievetsky, H. Ruegg, M. Santander, A. Stolin and V. Tolstoy for discussions. Two authors (J.L. and M.M.) would like to thank the University of Bordeaux I for the hospitality and financial support.

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